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# *On the Computation of Covariants by Transvection.\**

BY EMORY MCCLINTOCK.

Let  $A = A_0x^l + A_1x^{l-1}y + A_2x^{l-2}y^2 + \dots + A_ly^l$  and  $B = B_0x^m + B_1x^{m-1}y + \dots + B_my^m$  be covariants of the quantic  $U = ax^n + nbx^{n-1}y + \frac{1}{2}n(n-1)cx^{n-2}y^2 + \dots$ . If  $\alpha_x, \alpha_y$ , represent  $\frac{d}{dx}, \frac{d}{dy}$ , operating only on  $A$ , and  $\beta_x, \beta_y$ , represent  $\frac{d}{dx}, \frac{d}{dy}$ , operating only on  $B$ , then,  $k$  being any number not greater than either  $l$  or  $m$ ,

$$(A, B)_k = (\alpha_x\beta_y - \beta_x\alpha_y)^k AB, \quad (1)$$

where  $(A, B)_k$  represents the  $k^{\text{th}}$  transvectant of  $A$  and  $B$ , itself a third covariant, which let us denote by  $C = c_0x^{l+m-2k} + c_1x^{l+m-2k-1}y + \dots$ . That the effect of this operation of transvection ("ueberschiebung") is to produce a covariant is well known, as are also the theorems  $XA_0 = 0$ ,  $YA_h = (h+1)A_{h+1}$ , where  $X = a\frac{d}{db} + 2b\frac{d}{dc} + 3c\frac{d}{dd} + \dots$  and  $Y = nb\frac{d}{da} + (n-1)c\frac{d}{db} + \dots$ ; as well as the converse theorem that if  $X\phi = 0$ ,  $\phi$  is, if homogeneous, the leading coefficient of a covariant.

The actual performance of the operation indicated in (1) will effect the computation of  $C$  in all its details. It is, however, extremely troublesome. An easier method, no doubt usually followed, is to obtain by that means the leading coefficient only,  $c_0$ , and to derive  $c_1, c_2, \dots$  successively from  $c_0$ , expressed in terms of  $a, b, \dots$ , by repeated applications of the operation  $Y$ . For the first step I am accustomed to use the formula

$$c_0 = p^{(k)}A_0B_k - qp^{(k-1)}A_1B_{k-1} + q^2p^{(k-2)}A_2B_{k-2} + \dots \pm q^{(k)}A_kB_0, \quad (2)$$

where  $p = l - k + 1$ ,  $q = m - k + 1$ ,  $p^{(k)} = p(p+1)(p+2)\dots(p+k-1)$ . This will be found, on examination, to embody, omitting a common numerical

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factor, the results of the operation indicated in (1) for all terms independent of  $y$ . Something equivalent to (2) must no doubt be employed for the evaluation of  $c_0$  by every one who has to do with the computation of covariants by transvection. If we undertake to frame, in terms of  $A_0, A_1, \dots, B_0, \dots$ , corresponding formulæ for  $c_1, c_2$ , etc., we find them (except perhaps, as we shall see, when  $k = 1$ ) becoming too complex for use, and it is easier, as just stated, to derive the other coefficients directly from  $c_0$  by applying repeatedly the operation  $Y$ .

What cannot be done with formulæ, however, towards assigning the numerical coefficients resulting from (1), can be accomplished by a tabular construction, by the aid of which I find it easy to compute  $c_1, c_2$ , etc., by means of  $A_0, A_1, \dots, B_0, \dots$ , without performing the operation  $Y$ . It is perhaps better to present this method at first by a numerical example. Let it be required to calculate the third transvectant of two covariants of the fourth and sixth orders respectively. (This is the process recommended, for instance, by Clebsch and Gordan for producing degree 6, order 4, of the quintic.) Let us call the quartic  $A$ , and the sextic  $B$  (the choice is immaterial), and let us form a table having as arguments at the side the coefficients of  $A$  and at the top the coefficients of  $B$ , the spaces of the table to be occupied by the numerical multipliers, respectively, of the product of the argument at the side by the argument at the top, in the resulting expression for the transvectant desired, say  $C$ . We know that the latter can contain no

	$B_0$	$B_1$	$B_2$	$B_3$	$B_4$	$B_5$	$B_6$
$A_0$	0	0	0	3	12	30	60
$A_1$	0	0	—3	—6	—6	0	15
$A_2$	0	5	4	0	—4	—5	0
$A_3$	—15	0	6	6	3	0	0
$A_4$	—60	—30	—12	—3	0	0	0

expression  $A_g B_h$  in which  $g + h < k$ . All spaces for which the sum of the subscripts of the arguments is less than  $k$ , that is, less than 3 in this case, are therefore filled with zeros. We know that  $g + h = k$  for all terms composing  $c_0$ , that  $g + h = k + 1$  for all composing  $c_1$ , and so on. Therefore the diagonal  $A_0 B_3 \dots A_3 B_0$  will contain the multipliers for the terms of  $c_0$ , the next diagonal for those of  $c_1$ , and so on. We fill the diagonal  $A_0 B_3, A_1 B_2$ , etc., by means of formula (2), which in this case (where  $l = 4, m = 6, k = 3, p = 2, q = 4$ ),

dropping the common factor 8, gives  $c_0 = 3A_0B_3 - 3A_1B_2 + 5A_2B_1 - 15A_3B_0$ . These figures being inserted in the table, the remaining spaces in the line  $A_0$ , at top, are filled successively by multiplying the number to the left by the subscript of B for the space to be filled, and dividing the result in the first instance by 1, in the second by 2, and so on. The remaining spaces in the column  $B_0$  are completed in like manner, by multiplying the number above by the subscript of A, the rule for division being the same. The figures — 6, 4, 0, completing the diagonal of  $c_1$ , are now obtained by performing both operations, and dividing the sum by 1; those completing the diagonal of  $c_2$  are next computed likewise, dividing by 2, and so on. Thus, for  $A_1B_3$  in  $c_1$ , the number above (3) is multiplied by the subscript of A(1), and the number to the left (— 3) by the subscript of B(3), the results (3 and — 9) added together and divided by 1 giving — 6. Again, for  $A_3B_1$  in  $c_1$ , the number above (— 4) is multiplied by 3, and that to the left (6) by 4, producing  $(-12 + 24) \div 4 = 3$ . In all cases, the process to be pursued is embodied in the formula

$$x = (s\sigma + t\tau) \div r, \quad (3)$$

where  $x$  is the number required for the space  $A_sB_t$ ,  $\sigma$  that appearing in the space above,  $\tau$  that to the left, and  $r$  the number of the diagonal (counting the first as 0), or subscript of C. It is obvious that  $r = s + t - k$ .

In proof of (3), we observe that  $c_{r-1}$  consists of products having subscripts amounting in each case to  $s + t - 1$ , and including  $\sigma A_{s-1}B_t + \tau A_sB_{t-1}$ . The operation Y, which changes  $A_{g-1}$  into  $gA_g$ , will, if performed on those products, produce a succession of products having subscripts of the joint value of  $s + t$ , including among them the form  $A_sB_t$ . Since we have  $YA_{s-1} = sA_s$ ,  $YB_{t-1} = tB_t$ ,  $YC_{r-1} = rC_r$ , it follows that the coefficient of  $A_sB_t$  in  $YC_{r-1}$  is on the one hand equal to  $rx$ , and on the other hand equal to  $s\sigma + t\tau$ , whence (3) follows immediately. Since  $xc_0 = 0$ , when  $c_0$  is defined as in (2), showing that  $c_0$  is a new covariant, the original definition of transvection contained in (1) is not necessary, when the only object in view is the production of new covariants.

If we call the evaluation of (1) the first method of calculation, and the repeated use of Y upon the full expression of  $c_0$  in terms of  $a, b, \dots$  the second method, we may explain this third method as equivalent to the second, except that the actual performance of Y is avoided, while the work is assisted by recourse to the tabular form and to formula (3).

When  $k=1$  (the case of the Jacobian), the table takes a simple form, in that all rows and columns must be in arithmetical progression. Thus for  $l=4$ ,  $m=6$ ,  $k=1$ :

	<u>B<sub>0</sub></u>	<u>B<sub>1</sub></u>	<u>B<sub>2</sub></u>	<u>B<sub>3</sub></u>	<u>B<sub>4</sub></u>	<u>B<sub>5</sub></u>	<u>B<sub>6</sub></u>
A <sub>0</sub>	0	2	4	6	8	10	12
A <sub>1</sub>	— 3	— 1	1	3	5	7	9
A <sub>2</sub>	— 6	— 4	— 2	0	2	4	6
A <sub>3</sub>	— 9	— 7	— 5	— 3	— 1	1	3
A <sub>4</sub>	— 12	— 10	— 8	— 6	— 4	— 2	0

That this must be the case follows from (1), viz.,

$$(A, B) = \frac{dA}{dx} \frac{dB}{dy} - \frac{dB}{dx} \frac{dA}{dy}.$$

Here the coefficient of  $A_0B_g$  is  $lg$ ; that of  $A_1B_g$  is  $(l-1)g - (m-g) = lg - m$ ; that of  $A_2B_g$  is  $(l-2)g - 2(m-g) = lg - 2m$ ; and so on, with  $m$  as a constant difference. When  $k=1$ , therefore, we are able to lay down a sufficiently simple formula,

$$x = (l-s)t - s(m-t) = lt - ms. \quad (4)$$

This is so obvious, and the evaluation of Jacobians has been performed so often as compared with other transvectants, that it must have been observed repeatedly. The use of (4), however, is less advantageous than the tabular method with constant differences.

The operation (1) produces, as we have just seen in (4), a function of the subscripts of the first degree when  $k=1$ . Similarly, when  $k=2$ , it produces a function of the second degree, and so on. The rows and columns, therefore, when  $k=1$ , have constant first differences; when  $k=2$ , constant second differences, and so on. While this observation is chiefly serviceable when  $k=1$ , it will be found useful for other small values of  $k$ , particularly in complicated cases. For example, if  $l=4$ ,  $m=6$ ,  $k=2$ , the constant second difference of each row is 4, and of each column 10. From the method of forming the 0 row and 0 column it follows in all cases that the constant ( $k^{\text{th}}$ ) difference of each row is the first number in the 0 row, and that of each column the first number in the 0 column.

Since by interchanging  $x$  and  $y$  the whole process might be performed backwards, calling  $A_l A_0$ ,  $B_m B_0$ , etc., the numbers forming our table are always the same at opposite points, except that the sign may differ; that is to say, the multiplier for  $A_s B_t$  is numerically the same as for  $A_{l-s} B_{m-t}$ . This property, together with the constant  $k^{\text{th}}$  differences, makes the computation of the table a matter of little difficulty in most cases. It is not, indeed, really necessary to carry the table more than half way. When the evaluation of the resulting covariant has been effected as far as it can be by the help of half the table, the remaining complementary terms can be written off, as usual, by substituting — in the case of the sextic, for example —  $g$  for  $a$ ,  $f$  for  $b$ , and so on. Yet it is safer to carry the work out completely, and to check the results by verifying the complements.

When one of the covariants employed is the quantic itself, we may write  $a$  at the top of the first column,  $nb$  (say, for the sextic,  $6b$ ) at the top of the second, and so on, in lieu of  $B_0, B_1, \dots$ , or in addition to those headings. A better way, however, will be explained shortly. We may, of course, in simple cases, rearrange the whole table to suit our convenience, even substituting diagonals for columns, and *vice versa*. For example, I have used this table to compute the fourth transvectant of the covariant of the sextic of degree 2, order 4, with the sextic itself:

	For $c_0$ .	For $c_1$ .	For $c_2$ .
$ae - 4bd + 3c^2$	$e$	$2f$	$g$
$2af - 6be + 4cd$	$-d$	$-2e$	$-f$
$ag - 9ce + 8d^2$	$c$	$2d$	$e$
$2bg - 6cf + 4de$	$-b$	$-2c$	$-d$
$cg - 4df + 3e^2$	$a$	$2b$	$c$

Performing these multiplications, and dividing throughout by 2, we have

$$\begin{aligned} c_0 &= acg - 3adf + 2ae^2 - b^2g + 3bcf - bde - 3c^2e + 2cd^2, \\ c_1 &= -bcg - 8bdf + 9be^2 + 9c^2f - 17cde + adg + 8d^3 - aef, \\ c_2 &= aeg - 3bdg + 2c^2g - af^2 + 3bef - cdf - 3ce^2 + 2d^2e. \end{aligned}$$

We have here, as will be seen, the coefficients of the quadratic covariant of the third degree. This sort of table may be used to advantage whenever  $B =$  quantic and  $k =$  order of  $A$ .

I have, in the current number (January, 1892) of the *Bulletin of the New York Mathematical Society*, mentioned facts which have led me to believe that the simplest expression for any desired groundform, whose source is not that of a groundform of the quantic next lower, is that which is produced, when the operation is possible, by simple transvection from the nearest available covariant; using the phrase "simple transvection" as meaning "transvection with the quantic itself."

It will be found that for simple transvection the multipliers can be considerably reduced by incorporating with them the numbers at the head of the columns (coefficients of  $b, c, \dots$ ) and striking out common factors. The labor of doing so may, however, be obviated by applying formulæ different from, but derived from (2) and (3). Let  $s$ , as before, denote the row, and  $t$  the column,  $t$  being in this case the weight of that literal coefficient of the quantic which appears at the head of the column. On occasion, we may write  $a_0$  for  $a$ ,  $a_1$  for  $b$ , and so on. In this case  $B=U$ ,  $m=n$ ,  $q=n-k+1$ ,  $B_0=a$ ,  $B_1=nb$ ,  $B_2=\frac{1}{2}n(n-1)c$ , and so on. If now we reverse the order of the terms in (2), dividing throughout by  $\pm q^{(k)}$ , we have for this case

$$c_0 = A_k a - p A_{k-1} b + \frac{1}{2!} p^2 A_{k-2} c - \dots \quad (5)$$

Again, if we assume all other numbers in the table to be similarly divided, we have the right to employ (3) for the determination of the other numbers, and to modify it similarly by appropriate substitutions. In this way we obtain

$$z = (s\zeta + u\theta) \div r, \quad (6)$$

where  $z$  is the multiplier desired for  $A_s$ , column  $t$ ,  $\zeta$  the number above,  $\theta$  that to the left,  $r$  the same as before, and  $u = n - t + 1$ . It is to be observed that  $z$  and  $\zeta$  are respectively  $n(n-1) \dots (n-t+1)/t!$  times larger than  $s$  and  $\sigma$ , and that  $\theta$  is  $n(n-1) \dots (n-t)/t-1!$  times larger than  $r$ . We may say, therefore, that (6) is derived from (3) by making these substitutions and multiplying throughout by  $n(n-1) \dots (n-t+1)/t!$ , which leaves  $\theta$  with a coefficient  $t(n-t+1)/t = u$ . Or, we may obtain (6) *de novo* by remarking that  $Y_{A_{s-1}} = sA_s$ ,  $Y_{a_{t-1}} = (n-t+1)a^t$ ,  $Y_{c_{r-1}} = rc_r$ , so that the coefficient of  $A_s a_t$  in  $Y_{c_{r-1}}$  is on the one hand  $rz$ , on the other hand  $s\zeta + u\theta$ .

The formation of a table for simple transvection may be illustrated by the case  $n = 6$ ,  $l = 4$ ,  $k = 2$ ,  $p = 3$ :

	$\overset{0}{a}$	$\overset{1}{b}$	$\overset{2}{c}$	$\overset{3}{d}$	$\overset{4}{e}$	$\overset{5}{f}$	$\overset{6}{g}$
$A_0$	0	0	6	24	36	24	6
$A_1$	0	—3	—9	—6	6	9	3
$A_2$	1	0	—9	—16	—9	0	1
$A_3$	3	9	6	—6	—9	—3	0
$A_4$	6	24	36	24	6	0	0

The upper row, beginning with column  $k$ , consists of the binomial coefficients of the degree  $n - k$ , each multiplied by  $p^{(k)}$ ; the zero column is formed by writing 1 opposite  $A_k$ , then  $k + 1$ ,  $\frac{1}{2}(k + 1)^2$ ,  $\frac{1}{3!}(k + 1)^3$ , and so on; and the first diagonal, beginning with the space  $A_k a_0$ , consists of 1,  $p$ ,  $\frac{1}{2}p^2$ , and so on. The columns (alone) can be completed or verified by constant  $k^{\text{th}}$  differences. For another illustration, let us take a Jacobian,  $n = 6$ ,  $l = 4$ ,  $k = 1$ ,  $p = 4$ :

	$\overset{0}{a}$	$\overset{1}{b}$	$\overset{2}{c}$	$\overset{3}{d}$	$\overset{4}{e}$	$\overset{5}{f}$	$\overset{6}{g}$
$A_0$	0	—4	—20	—40	—40	—20	—4
$A_1$	1	2	—5	—20	—25	—14	—3
$A_2$	2	8	10	0	—10	—8	—2
$A_3$	3	14	25	20	5	—2	—1
$A_4$	4	20	40	40	20	4	0

That case of simple transvection in which  $k = l$ , so that  $p = 1$ , supplies a numerical table which may be written down without calculation, and the table may be simplified further by substituting columns for diagonals. (An instance has already been shown.) Under the general rule, when  $n = 6$ ,  $l = 4$ ,  $k = 4$ ,  $p = 1$ , we should put the table in this form:

	$\overset{0}{a}$	$\overset{1}{b}$	$\overset{2}{c}$	$\overset{3}{d}$	$\overset{4}{e}$	$\overset{5}{f}$	$\overset{6}{g}$
$A_0$	0	0	0	0	1	2	1
$A_1$	0	0	0	—1	—2	—1	0
$A_2$	0	0	1	2	1	0	0
$A_3$	0	—1	—2	—1	0	0	0
$A_4$	1	2	1	0	0	0	0

The simplification now in question consists in reducing the number of columns to  $n - l + 1$ , one column for each coefficient of  $C$  in the result; in writing the binomial coefficients for the degree  $n - l$  at the head of the columns; in affecting  $A_1, A_3, \dots$ , having odd subscripts, with the negative sign; in writing  $a$  at the foot of the first column,  $b$  above it, and so on, the rows being then completed in alphabetical order. The illustrative table just given then takes this shape :

	For $c_0$ $\times 1$	For $c_1$ $\times 2$	For $c_2$ $\times 1$
	<hr/>	<hr/>	<hr/>
$A_0$	$e$	$f$	$g$
$-A_1$	$d$	$e$	$f$
$A_2$	$c$	$d$	$e$
$-A_3$	$b$	$c$	$d$
$A_4$	$a$	$b$	$c$

The coefficient of  $c_0$  when  $p = 1$  is, by (5),  $A_k a - A_{k-1} b + A_{k-2} c - \dots$ , the numerical multipliers being equal, with alternate signs. But by (6), when the numerical multipliers, say for  $c_{r-1}$ , are equal with alternate signs, we have  $\theta = -\zeta$ ,  $rz = \theta(u - s) = \theta(n - t - s + 1) = \theta(n - l + 1 - r)$ . If  $n - l$ , which is the order of  $C$ , be represented by  $\lambda$ , we find thus that the numerical value of each multiplier for  $c_0$  is 1, for  $c_1$  is  $\lambda$ , for  $c_2$  is  $\frac{1}{2}\lambda(\lambda - 1)$ , and so on according to the binomial series, the multipliers of each column having alternate, and of each row the same, signs, as constructed in the foregoing table.